Floer homology

Say something on Arnold's conjecture.

Morse	Floer
(1) smooth Manifold X	(1) Fix a cpct symplectic mfd (M,ω). Space of contractible loops $\mathcal{L}M = \{\gamma : S^1 \to M \ C^\infty \ \underline{\text{contractible}}\}$ i.e. $\exists\, u : D^2 \to M\ C^\infty$ s.t. $u_{\mid S^1} = \gamma$. $\boxed{Q1}$ Smooth structure?
(2) Morse function $f : X \to \mathbb{R}$	(2) Action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ associated to non-degenerate 1-per. Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$ C^∞. $\boxed{\pi_2(M) = 0}$ $$\mathcal{A}_H(\gamma) = -\int_{D^2} u^*\omega + \int_0^1 H(t, \gamma(t))\,dt.$$ $\boxed{Q2}$ Does not depend on u?
(3) Critical points of f : $\mathrm{Crit}(f)$	(3) Prop3 Critical points of $\mathcal{A}_H$ are 1-periodic contractible orbits of X_H. $(\omega(X_{H_t}, \cdot) = -dH)$.
(4) Index of $p \in \mathrm{Crit}(f)$	(4) Conley-Zehnder index $i_{CZ}(\gamma)$ of $\gamma \in \mathrm{Crit}(\mathcal{A}_H)$. $\boxed{Q4}$ Definition?
(5) Choose a metric g on M	(5) Choose an almost complex structure J on TM. $\boxed{Q5}$ Definition. $\to$ Induces a metric on $\mathcal{L}M$. $\boxed{Q6}$ How? $\boxed{Q8}$ Compute $\nabla\mathcal{A}_H$ for this metric.
(6) Flow lines of $-\nabla f$. $\mathcal{N} = \{\gamma : \mathbb{R} \to X \ : \ \frac{d\gamma}{dt}(t) = -\nabla_{\gamma(t)} f\}$.	(6) Flow lines of $-\nabla\mathcal{A}_H$: $(\mathbb{R} \to \mathcal{L}M)$ $u : \mathbb{R}\times S^1 \to M$ $$\frac{\partial u}{\partial s} = -\nabla\mathcal{A}_H(u(s,\cdot))$$ Q7 $\Rightarrow$ Floer equation (FE) $\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \nabla_t H(u) = 0$

Lem: X cpt $\Rightarrow \begin{cases} \lim_{-\infty} \gamma = p \\ \lim_{+\infty} \gamma = q \end{cases}$ $p, q \in \mathrm{Crit}(f)$

(7) Moduli space of flow lines

$$\mathcal{N}(p,q) = \{\gamma \in \mathcal{N} : \lim_{t\to-\infty} \gamma(t) = p \quad \lim_{t\to+\infty} \gamma(t) = q\}.$$

$$\mathcal{M}(p,q) = \mathcal{N}(p,q)/_{\mathbb{R}} \text{ translation}$$

Assumption: The pair (f, g) should satisfy the <u>Smale condition</u>.

Prop: If (f,g) is Smale, then $\mathcal{M}(p,q)$ is a submanifold of X of dim $i(p) - i(q) - 1$.
Moreover, it can be compactified to become a compact submanifold with corners and bdry:

$$\bigcup_{p_i \in \mathrm{Crit}(f)} \underbrace{\mathcal{M}(p,p_1) \times \cdots \times \mathcal{M}(p_\ell, q)}_{\text{"broken lines"}}$$

Cor: If (f,g) is Smale and $i(p) = i(q) + 1$, then $\mathcal{M}(p,q)$ is finite.

Def: $\mathcal{N} = \{ u : \mathbb{S}^1 \times \mathbb{R} \to M$ sol of (FE) such that $E(u) := \int_{\mathbb{S}^1\times\mathbb{R}} \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt < +\infty \}$

Lem: For $u \in \mathcal{N}(x,y)$, $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$

Thm: Recall M is aspherical and H non-degenerate.
Then for $u \in \mathcal{N}$,

$$\begin{cases} \lim_{s\to-\infty} u = x \\ \lim_{s\to+\infty} u = y \end{cases} \quad \text{for some } x, y \in \mathrm{Crit}(\mathcal{A}_H)$$

Proof: elliptic regularity sol in $W^{1,p}_{loc}$ $p > 2 \Rightarrow C^\infty$.

(7) Moduli spaces of Floer solutions

$$\mathcal{N}(x,y) = \{ u \in \mathcal{N} : \lim_{s\to-\infty} u = x, \lim_{s\to+\infty} u = y \}.$$
u sol. of (FE),

$$\mathcal{M}(x,y) = \mathcal{N}(x,y)/_{\mathbb{R}} \text{ translation on } s \text{ variable}$$

Assumption: (H, J) should be <u>regular</u>.

differential of Floer operator (Fredholm) at $u \in \mathcal{N}$
$dF_u : \mathcal{B}^{1,p}(x,y) \to L^p(\mathbb{R}\times\mathbb{S}^1, TM)$ is surjective
variété Banach modeled on $W^{1,p}(\mathbb{R}\times\mathbb{S}^1, TM)$.

Prop: If (H,J) is regular, then . . .
. . .

Cor: Idem

Cor: If (f,g) is Smale and $i(p)=i(q)+2$, then $\overline{\mathcal{M}(p,q)}$ cpct manifold with corners and bdry:

$$\bigcup_{\substack{i(c)=i(p)-1 \\ c\in \mathrm{Crit}(f)}} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$$

(8) Morse complex

(9) Morse homology

Cor: Idem

(8) Floer complex

$$CF_k^{\lambda}(H,J) = \langle x\in \mathrm{Crit}(\mathcal{A}_H) \mid i_{CZ}(x)=k\rangle_{\mathbb{Z}_2}$$

$$\partial_k x = \sum_{y\in \mathrm{Crit}(\mathcal{A}_H)} m(x,y)\, y$$

where $m(x,y) = \#\mathcal{M}(x,y)$.

Cor: $\partial_{k-1}\circ\partial_k = 0$.

Proof: $\partial_{k-1}\circ\partial_k(x) = \sum_{i(z)=k-2} z\cdot \underbrace{\left(\sum_{i(y)=k-1} m(x,y)m(y,z)\right)}_{\shortparallel}$

$\# \left(\partial\, \underline{\overline{\mathcal{M}(x,z)}}\right)$ 1-dim mfld with corners and bdry.

$\shortparallel$

$0 \mod 2$.

$\boxtimes$

(9) Floer homology

$$HF_k(H,J) = \mathrm{Ker}(\partial_k)/\mathrm{Im}(\partial_{k+1})$$

Thm: HF_k does not depend on the regular pair (H,J).

Thm: If H is an autonomous Morse function C^2 small enough, then for generic J, (H,J) is regular, $(H, \underset{g}{\underbrace{\omega(\cdot, J\cdot)}})$ is Smale and

$$CF_*(H,J) = CM_{*+m}(H,g).$$

Proof: is involved. Floer and Morse flow lines coincide.

NB: When H is C^2 small enough, then $\mathrm{crit}(\mathcal{A}_H) = \mathrm{crit}(H)$.

Proof: TODO ?.

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(10) Filtered Hamiltonian Floer homology

$$CF_k^\lambda(H,J) = \langle x \in \mathrm{Crit}(\mathcal{A}_H) \mid i_{CZ}(x) = k, \ \mathcal{A}_H(x) < \lambda \rangle_{\mathbb{Z}_2}$$

∂_k induces $\partial_k : CF_k^\lambda \to CF_{k-1}^\lambda$

$$HF_k^\lambda(H,J) = H_k(CF_*^\lambda).$$

$\boxed{Q8}$ Dependence in (H,J) ?   TODO: ideas on how it works ?.

Thm: (Schwarz 2000) Let $H : S^1 \times M \to \mathbb{R}$ be Hamiltonian.

Assume H is normalised: $\int_M H(t,\cdot)\, \omega^n = 0 \quad \forall t \in S^1$.

Then, $HF_k^\lambda(H,J)$ only depends on $\phi = \phi_H^1$ (time 1 of Hamil. flow ϕ_H^t of X_H.)

denoted $HF_k^\lambda(\phi)$

Def: (Hamiltonian diffeo)

$$\mathrm{Ham}(M,\omega) = \{\phi \in \mathrm{Diffeo}(M),\ \phi = \phi_H^1,\ H \text{ Hamiltonian}\}.$$

Cor: $\phi \in \mathrm{Ham}(M,\omega) \mapsto \mathcal{B}_k(\phi) =$ barcode of $HF_k^\bullet(\phi)$
(up to a shift)

II. Conley-Zehnder index

H non-degenerate $\qquad x \in \mathrm{Crit}(\mathcal{A}_H)$. 1-periodic contractible orbit of X_H. $\left(\begin{array}{c} \omega(X,Y) \\ \shortparallel \\ \omega(AX,AY) \end{array}\right)$

(1) Associate a path $t \in [0,1] \mapsto A(t) \in Sp(2n) = \{A \ 2n \times 2n \text{ matrices s.t. } A^T \Omega A = \Omega\}$ $\Updownarrow$
where $\Omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$

with $A(0) = I_n$ and $1 \notin \mathrm{spec}(A(1))$ that is uniquely defined up to htpy.

(2) Associate to any htpy class of such paths an integer $\mu \in \mathbb{Z}$, via a map $\rho: Sp(2n, \mathbb{R}) \to S^1$ inducing an isomorphism $\tilde{\rho}: \pi_1(Sp(2n,\mathbb{R})) \xrightarrow{\sim} \pi_1(S^1) \cong \mathbb{Z}$.

Step (1): Choose $u: D^2 \to M$ s.t. $u_{|S^1} = x$.

Since D^2 is contractible, $u^* TM$ is trivial and any two trivializations are homotopic. $\quad u^*TM \cong D^2 \times \mathbb{R}^{2n}$

Linearized flow $d\phi_H^t(x(0))$:
$$\begin{array}{ccc} T_{x(0)}M & \longrightarrow & T_{x(t)}M \\ \wr\shortparallel & & \wr\shortparallel \\ \{0\} \times \mathbb{R}^{2n} & \xrightarrow{A(t)} & \{t\} \times \mathbb{R}^{2n} \\ \text{\scriptsize $S^1 \subseteq D^2$} & & \text{\scriptsize $S^1 \subseteq D^2$} \end{array}$$
with $A(t) \in Sp(2n)$.

- $A(0) = I_n$ because $d\phi_H^0 = id_{T_{x(0)}M}$
- $A(1) = d\phi_H^1$ so $1 \notin \mathrm{spec}(A(1))$ (H non-degenerate)

$$Sp^*(2n) = \{A \in Sp(2n) \mid \det(A - I_n) \neq 0\}.$$

$$A \in \mathcal{S} = \{\gamma: [0,1] \to Sp(2n) \mid \gamma(0) = I_n,\ \gamma(1) \in Sp^*(2n)\}.$$

- <u>htpy invariance</u>?

other choice of u: $v: D^2 \to M \quad w_{|S^1} = x$. $\quad A(t) \to A'(t)$

TODO: explain trivialization and homotopies.

then $w = u \# v : S^2 \to M$ gluing u and v. $\quad \to A''(t)$

Since $\pi_2(M) = 0$, w^*TM is also trivial and any two triv. are htpic.

Therefore $A(.)$, $A'(.)$, $A''(.)$ are homotopic in $\mathcal{S}$

<u>CCl</u>: $t \mapsto A_x(t)$ induces a well-defined htpy class of maps $[0,1] \to Sp^*(2n)$.

Step (2): (a) $\pi_1(Sp(2n)) \cong \mathbb{Z}$. Induced by $\rho : Sp(2n) \to S^1$

Proof: If $A \in Sp(2n)$ then $S = \sqrt{AA^T}$ symmetric positive definite and symplectic.

So that $B = AS^{-1}$ is symplectic and orthogonal

$$U(n) = Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$$
$$\parallel$$
$$\{ C \in \mathcal{M}_n(\mathbb{C}) : C^{-1} = \bar{C}^T \}.$$

indeed: standard hermitian form: in coordinates $z = x + iy$

$$\langle (x, y), (x', y') \rangle = xx' + yy' + i(yx' - xy')$$

So $Sp(2n) \underset{\text{homeo}}{\simeq} U(n) \times \mathcal{Y}$ ← symplectic symm. positive-definite

Prop: $\det_{\mathbb{C}} : U(n) \to S^1$ induces an iso $\pi_1(U(n)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$.

Exo: $\mathcal{Y}$ is contractible.

Cor:

$$\rho : \begin{array}{ccc} Sp(2n) & \longrightarrow & S^1 \\ A & \longmapsto & \det_{\mathbb{C}}\left(A \left(\sqrt{AA^T}\right)^{-1}\right) \end{array}$$

induces iso: $\pi_1(Sp(2n)) \cong \mathbb{Z}$.

(b) "From $t \mapsto A(t)$ to $i_{CZ}(x)$"

Lem: (Conley - Zehnder)

Sp^* has two connected components $Sp^+ = \{A \in Sp(2n) : \det(A - I_n) > 0\}$

$Sp^- = \{ \text{―――――――――} < 0 \}$.

and the inclusions $Sp^{\pm} \hookrightarrow Sp(2n)$ induce zero morphisms on f^{al} group.

Choose then $W^+ = -I_n \in Sp^+$ $\quad W^- = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & 0 \\ 0 & -I_n \end{pmatrix} \in Sp^-$

and for all $A \in Sp^{\pm}$ a path α_A from A to the corresponding $W^{\pm}$ in $Sp^{\pm}$.

Def: $\gamma : [0,1] \longrightarrow Sp(2n, \mathbb{R})$ $\quad \gamma(0) = I_{2n}$ $\quad \gamma(1) = A \in Sp^*$.

Let $\hat{\gamma} = \gamma * \alpha_A$ be the concatenation.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp(it)"] \\ {[0,1]} \arrow[ur, "\Delta"] \arrow[r, "\rho \circ \hat{\gamma}"'] & S^1 \end{tikzcd}$$

$$\mu(\gamma) = \frac{1}{\pi} \left(\Delta(1) - \Delta(0) \right)$$

Prop:

1) $\mu(\gamma) \in \mathbb{Z}$

2) Two paths γ, γ' with $\gamma(0) = \gamma'(0) = I_n$ and $\gamma(1) = \gamma'(1) = A \in Sp^*$ are htpic among such paths iff $\mu(\gamma) = \mu(\gamma')$.

3) sign of $\det(A - I_n)$ is $(-1)^{\mu(\gamma) - n}$

case of C^2 small autonomous flows fn H

4) If S is symmetric invertible with $\|S\| < 2\pi$ and $\gamma(t) = \exp(tJS)$, then $\mu(\gamma) = \mathrm{ind}(S) - n$ where $\mathrm{ind}(S) = \#$ negative eigenvalues.

Def: $x \in \mathrm{Crit}(A_H)$ $\quad i_{CZ}(x) = \mu(\gamma)$ $\quad$ where $\gamma : t \mapsto A(t)$ constructed above.

Answer to questions:

$\boxed{Q1}$ Choose metric g on M.

$\exp : TM \to M$

$(x,v) \to \exp_x(v)$ ← time 1 of unique geodesic starting from x with velocity v

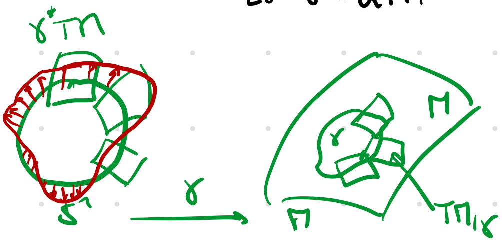
Recall: $\exp_x : (T_xM, 0) \to (M, x)$ local diffeo

Let $\gamma \in \mathcal{L}M$.

$$\begin{tikzcd} \gamma^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\ S^1 \arrow[r, "\gamma"] & M \end{tikzcd}$$

$\gamma^*TM = \{(t, x, v) \in S^1 \times TM : \gamma(t) = x\}$.

$\Gamma(\gamma^*TM) = \{Y : S^1 \to \gamma^*TM : \pi \circ Y = \mathrm{Id}_{S^1}\}$

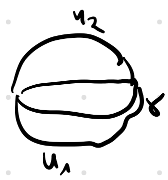


$\exp_\gamma$: small $Y \in \Gamma(\gamma^*TM) \mapsto \Big(\exp_\gamma Y : \underbrace{\overset{S^1 \to M}{t \mapsto \exp_{\gamma(t)} Y(t)}}_{C^\infty(S^1, M)}\Big) \in$ neigh. of γ in $\mathcal{L}M$.

local charts for $\mathcal{L}M$.

and : $T_\gamma \mathcal{L}M \simeq \Gamma(\gamma^*TM)$

$\boxed{Q2}$ $\mathcal{A}(\gamma) = -\int_{D^2} u^*\omega$ does not depend on u when $\pi_2(M) = 0$.



If $u_1, u_2 : D^2 \to M$ s.t. $u_i|_{S^1} = \gamma$. then $u_1 \# u_2 : S^2 \to M$ htpic cst.

$$\mathcal{A}_1(\gamma) - \mathcal{A}_2(\gamma) = \int_{S^2} (u_1 \# u_2)^*\omega \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \int_{B^3} d(u_1 \# u_2)^*\omega = \int_{B^3} (u_1 \# u_2)^* \underbrace{d\omega}_{=0} = 0$$

Prop 3: Critical points of $\mathcal{A}_H$ are 1-periodic contractible orbits of X_H.